Trapped surfaces in prolate collapse in the Gibbons-Penrose construction

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February 7, 2008

Abstract

We investigate existence and properties of trapped surfaces in two models of collapsing null dust shells within the Gibbons-Penrose construction. In the first model, the shell is initially a prolate spheroid, and the resulting singularity forms at the ends first (relative to a natural time slicing by flat hyperplanes), in analogy with behavior found in certain prolate collapse examples considered by Shapiro and Teukolsky. We give an explicit example in which trapped surfaces are present on the shell, but none exist prior to the last flat slice, thereby explicitly showing that the absence of trapped surfaces on a particular, natural slicing does not imply an absence of trapped surfaces in the spacetime. We then examine a model considered by Barrabès, Israel and Letelier (BIL) of a cylindrical shell of mass M and length L, with hemispherical endcaps of mass m. We obtain a "phase diagram" for the presence of trapped surfaces on the shell with respect to essential parameters $\lambda \equiv M/L$ and $\mu \equiv m/M$. It is found that no trapped surfaces are present on the shell when λ or μ are sufficiently small. (We are able only to search for trapped surfaces lying on the shell itself.) In the limit $\lambda \to 0$, the existence or nonexistence of trapped surfaces lying within the shell is seen to be in remarkably good accord with the hoop conjecture.

1 Introduction

It is well known from the singularity theorems [1] that singularities must occur in a wide variety of circumstances relevant to gravitational collapse—in particular, when trapped surfaces or reconverging light cones [2] are present. However, there is still little direct evidence on whether cosmic censorship holds, *i.e.*, whether the singularities of gravitational collapse are always hidden in black holes (see [2] and

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[3] for recent reviews). Much of our knowledge of what occurs in gravitational collapse is based upon the analysis of spherically symmetric models and their linearized perturbations. Without results from other models, we cannot know which features might be shared by—and which absent from—general kinds of collapse.

It has been suggested by a number of authors that *highly prolate* gravitational collapse may be of a qualitatively different character from spherical collapse (see, *e.g.* [4]). An explicit statement of this view has been expressed in the "hoop conjecture" [5]:

> Horizons form when and only when a mass M gets compacted into a region whose circumference in EVERY direction is $C \lesssim 4\pi M$.

The intended interpretation of this statement is that a distribution of matter that is elongated in one dimension but is sufficiently narrow in the other dimensions might be expected to collapse without the creation of a event horizon, so that any singularity that formed would be "naked". However, there are serious difficulties in giving a precise statement of the hoop conjecture, both in defining the "mass"—one would want to include concentrations of gravitational energy, for which there is no local definition—and the "circumference"—which always can be made arbitrarily small if no choice of spacelike slicing is specified.

Nevertheless, numerical simulations by Shapiro and Teukolsky [6] of the collapse of a collisionless gas cloud appeared to lend support to the hoop conjecture. For a sufficiently prolate cloud, a singularity was found to form (after which point the numerical evolution could not be continued) but no trapped surfaces were found prior to the singularity on the family of maximal hypersurfaces they used for their time evolution.

Barrabès, Israel and Letelier [7] considered a simple analytic counterpart of the Shapiro-Teukolsky examples within the Gibbons-Penrose construction (see below). This model consists of an imploding shell of photons (null dust) in the shape of a cylinder with hemispherical endcaps and with piecewise uniform mass-energy density. They found, similarly, that given a fixed amount of matter, if the cylindrical portion of the shell is made long enough (*i.e.* sufficiently prolate), the 2-surfaces formed by the intersection of the shell with flat hyperplanes of the Minkowski region interior to the shell are never trapped at any stage of collapse.

However, in both the numerical and analytic models, the conclusion that no trapped surfaces exist relies on a specific choice of time slicing. Indeed, as a counterargument to the Shapiro-Teukolsky result, Wald and Iyer [8] proved that even in Schwarzschild spacetime, it is possible to choose a (highly non-spherical) time slice which comes arbitrarily close to the singularity, yet for which no trapped surfaces are found to its past. This example explicitly demonstrates that the absence of trapped surfaces in a particular time slicing does not establish the absence of trapped surfaces in the spacetime. Nevertheless, the slicing in [8] is rather contrived, and it is far from clear that anything similar to this is occurring in the Shapiro-Teukolsky examples.

The main purpose of this paper is to investigate the existence and properties of trapped surfaces in highly prolate collapse in the context of spacetime models produced by the Gibbons-Penrose construction. The Gibbons-Penrose spacetimes contain imploding shells of null dust, with the region of spacetime interior to the shell being flat. These models have the advantage that it is relatively easy to search for trapped surfaces lying on (or, more precisely, infinitesimally outside of) the collapsing shell. Unfortunately, in these models, the spacetime outside of the shell is not explicitly known, so one cannot determine the existence or properties of trapped surfaces lying at a finite distance outside of the shell.

In section 2, we review the Gibbons-Penrose construction. In section 3, we examine the collapse of an initially prolate spheroidal shell. We obtain an explicit example—previously suggested in [9]—in which, in the natural flat hyperplane time slicing in the region interior to the shell, the singularity forms at the poles before any trapped surface has finished forming along the equator. Thus, a numerical code using this natural time coordinate—which would halt at the time the singularity begins to form—would not find any trapped surfaces on the shell. Nevertheless, in the fully evolved spacetime, trapped surfaces do occur on the shell. Thus, this example explicitly displays the behavior closely analogous to that constructed in [8], but does so in the context of prolate collapse with a natural choice of time slicing.

In section 4, we re-examine the Barrabès-Israel-Letelier (BIL) model of a collapsing cylindrical shell with hemispherical endcaps, in the case where $\lambda = M/L < 1/8$, where M is the mass of the cylinder and L its length. BIL showed that in this case, none of the 2-surfaces formed by the intersection of the shell with the flat hyperplanes of the interior region are trapped surfaces. Here, we search for trapped surfaces on the shell which do not lie in these hyperplanes. We present a "phase diagram" for the the presence or absence of trapped surfaces as a function of the essential scale-invariant parameters λ and $\mu \equiv m/M$, where m the mass of each endcap. We find that no trapped surfaces are present on the shell for λ or μ sufficiently small (although we expect that trapped surfaces will be present outside of the shell in all cases). We find that the existence or nonexistence of trapped surfaces lying on the shell is in remarkably good agreement with what would be expected from the hoop conjecture.

2 The Gibbons-Penrose construction

The Gibbons-Penrose construction considers a thin shell of matter collapsing inward at the speed of light from past null infinity. The trajectory of the shell describes a null hypersurface $\mathcal{N}$ which divides the spacetime into two regions: the interior, which is Minkowski, and the exterior, which is not flat and generally contains gravitational radiation. Singularities form where the null hypersurface develops caustics.

The construction is completely specified by two functions, one defining the shape of the collapsing shell, and one defining its (distributional) mass density.

The degrees of freedom associated with the shape correspond to a choice of 2-surface $\mathcal{S}(T)$ in the hyperplane $t = T$ of Minkowski spacetime. The hypersurface $\mathcal{N}$ is then generated by the ingoing normal null geodesics through $\mathcal{S}(T)$. We restrict to the case where $\mathcal{S}(T)$ is convex, so that no caustics appear to the past of our initial time. The "initial" mass density $\sigma(T)$ can be chosen to be any non-negative function on $\mathcal{S}(T)$. The mass density is then extended to the whole of $\mathcal{N}$ by the conservation equation.

One can then, in principle, go about searching for trapped surfaces in the spacetime. In general, this, of course, requires solving Einstein's equations in the region of interest. The great advantage of the Gibbons-Penrose construction is that as long as one focuses attention on surfaces infinitesimally outside $\mathcal{N}$, one need not know anything about the spacetime outside $\mathcal{N}$; that is, given some candidate compact 2-surface in $\mathcal{N}$, we can determine whether it is trapped without ever solving for the exterior spacetime. The expansion θ of future-directed, outgoing null geodesics normal to the 2-surface, just before they cross the shell, is [10]

$$\theta_{\text{inside}} = -\text{Tr } L \tag{1}$$

where L is the extrinsic curvature of the 2-surface under its embedding in Minkowksi spacetime, with respect to outgoing null normals. Just outside the shell, we have

$$\theta_{\text{outside}} = \theta_{\text{inside}} + (\Delta\theta)_{\text{matter}} \tag{2}$$

where, by integrating the Raychaudhuri equation [11] across the shell (so that only the distributional source term contributes), we find that

$$(\Delta\theta)_{\text{matter}} = -8\pi\sigma \tag{3}$$

Thus, the 2-surface is trapped if

$$-\text{ Tr } L < 8\pi\sigma \tag{4}$$

everywhere on the surface.

The project is further simplified if one looks instead for an *outer marginally-trapped surface* (MTS), *i.e.* a compact 2-surface for which the expansion of future-directed outgoing null normals is everywhere zero. By an argument similar to one given in [1] for trapped surfaces lying in a spacelike hypersurface, if $\mathcal{N}$ contains trapped surfaces, then the outer boundary of the region containing the trapped surfaces is an MTS. The absence of an outer marginally-trapped surface on $\mathcal{N}$ thus implies the absence of trapped surfaces on $\mathcal{N}$.

The advantage of searching for MTS's is that a MTS satisfies the equation

$$-\text{ Tr } L = 8\pi\sigma \tag{5}$$

rather than an inequality. Thus, rather than attempt to explore the myriad of possibilities for where trapped surfaces might occur, one studies eq. (5) and attempts to determine if any global solutions exist.

In the following, we narrow our search for trapped surfaces in the whole spacetime to the search for MTS's confined to the null hypersurface $\mathcal{N}$.

3 The collapsing spheroid

We now consider the collapse of a prolate ellipsoid in the Gibbons-Penrose construction. We choose Minkowski coordinates (t, x, y, z) in what will be the interior of $\mathcal{N}$, and specify the initial shape $\mathcal{S}_0$ at $t = 0$ to be an axisymmetric ellipsoid (*i.e.*, a spheroid)

$$x^\mu(\theta, \phi) = (0, a\sin\theta\cos\phi, a\sin\theta\sin\phi, b\cos\theta) \tag{6}$$

$$\theta \in [0, \pi], \phi \in [0, 2\pi) \tag{7}$$

where $a < b$ so that the spheroid is prolate. Following the ingoing null normals forward and backward in time yields the form of $\mathcal{N}$,

$$x^\mu(t, \theta, \phi) = \left(t, \left(a - \frac{bt}{\sqrt{E}}\right)\sin\theta\cos\phi, \left(a - \frac{bt}{\sqrt{E}}\right)\sin\theta\sin\phi, \left(b - \frac{at}{\sqrt{E}}\right)\cos\theta\right) \tag{8}$$

where

$$E(\theta) = a^2\cos^2\theta + b^2\sin^2\theta \tag{9}$$

The metric induced on the intersection of $\mathcal{N}$ with the hyperplanes of constant t is

$$ds^2 = \left(1 - \frac{abt}{E^{3/2}}\right)^2 E d\theta^2 + a^2\left(1 - \frac{bt}{aE^{1/2}}\right)^2 \sin^2\theta d\phi^2 \tag{10}$$

This metric becomes truly singular when the quantities in parentheses become zero. This is just the condition that t is equal to a principal radius of curvature of the initial spheroid, where these radii of curvature are given by

$$R_1^{-1} = \kappa_1(\theta) = \frac{ab}{E^{3/2}} \tag{11}$$

$$R_2^{-1} = \kappa_2(\theta) = \frac{b}{aE^{1/2}} \tag{12}$$

Since $\kappa_1 \leq \kappa_2$ (with equality only at the poles), the singularity actually occurs at $t(\theta) = 1/\kappa_2(\theta)$.

The evolution of the initial mass density σ_0 (which so far is unspecified) is governed by the local mass-conservation equation

$$\frac{\partial}{\partial t}(\sigma dA) = 0 \tag{13}$$

where dA is the area form obtained from the metric (10),

$$dA = aUV\sqrt{E}\sin\theta\ d\theta \wedge d\phi \tag{14}$$

where

$$U(\theta, t) = 1 - \kappa_2 t \tag{15}$$

$$V(\theta, t) = 1 - \kappa_1 t \tag{16}$$

Given some σ_0, then,

$$\sigma(t,\theta,\phi) = \frac{\sigma_0}{UV} \tag{17}$$

Equations (8) and (17) define the Gibbons-Penrose construction for the collapsing spheroid.

Now consider an arbitrary spacelike 2-surface on $\mathcal{N}$,

$$t = F(\theta,\phi) \tag{18}$$

We seek the condition on F such that this surface is marginally-trapped. The embedding of the 2-surface in Minkowski spacetime is given by equations (18) and (8), and its extrinsic curvature is readily calculated. Combining this result with (17), we find that eq. (5) becomes

$$\nabla^2 F + \frac{1}{2}\left(\frac{\kappa_1}{V} + \frac{\kappa_2}{U}\right)(|\nabla F|^2 + 1) = 8\pi\frac{\sigma_0}{UV} \tag{19}$$

where Laplacians and gradients are with respect to the metric (10) and the functions U and V are to be evaluated on the 2-surface (*i.e.* replace t by $F(\theta,\phi)$ everywhere).

Finally, we will restrict attention to axisymmetric solutions $F = F(\theta)$ since the outer boundary of the region containing trapped surfaces must have the same symmetries as the spacetime itself. In this case eq. (19) reduces to

$$\begin{aligned}\frac{U}{V\sqrt{E}\sin\theta}\frac{d}{d\theta}\left(\frac{\sin\theta}{\sqrt{E}}F'\right) + (3\kappa_1 - \kappa_2 - 2\kappa_1\kappa_2 F)\frac{FF'}{2EV^2}\left(\frac{F'}{F} - \frac{E'}{E}\right)& \\ +\frac{1}{2}(\kappa_1 + \kappa_2 - 2\kappa_1\kappa_2 F) = 8\pi\sigma_0&\end{aligned} \tag{20}$$

The parameters and free functions of this model can be made to yield a spacetime where the outermost marginally-trapped surface encloses the singularity, but forms later at the equator than the singularity does at the poles, in the time coordinate t. A numerical code using this time coordinate would then halt before any trapped surfaces have formed. To obtain an explicit example where this occurs, we will specify a form of F, and use (20) to determine the requisite initial mass density, which we must verify to be everywhere positive and finite. A relatively simple choice of MTS is the surface of constant V

$$F(\theta) = \frac{\gamma a^2}{b^2\kappa_1} \tag{21}$$

where γ is some constant. In order that the MTS have the special properties described above, we must restrict γ according to

$$\frac{a}{b} < \gamma < 1 \tag{22}$$

Substituting F into (20) gives the initial mass density,

$$\sigma_0 = \frac{\kappa_1}{8\pi}(A\cos^6\theta + B\cos^4\theta + C\cos^2\theta + D) \tag{23}$$

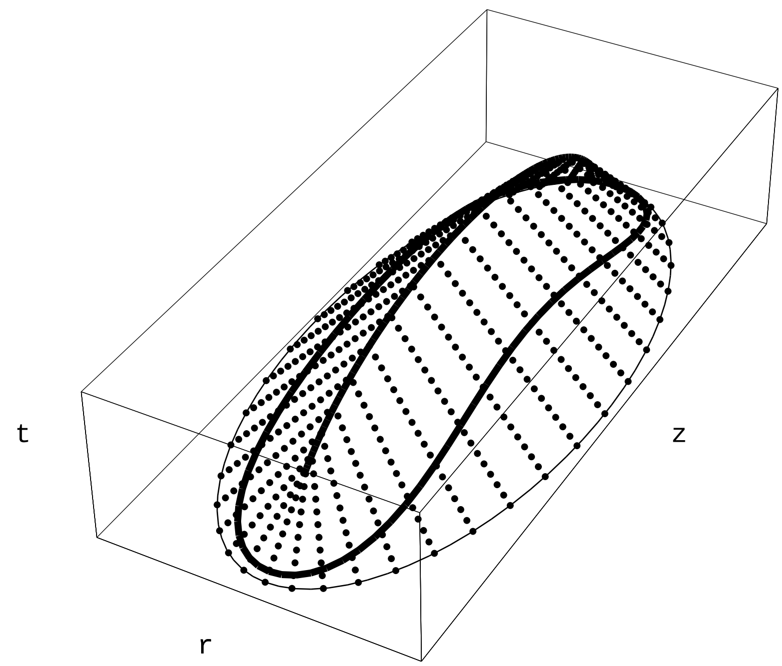

Figure 1: The chosen MTS ($\gamma = 1/2,\ a = 1,\ b = \sqrt{5}$) and the singularity ($\phi$-coordinate suppressed). Dotted lines indicate null generators of the hypersurface $\mathcal{N}$. Note that the MTS forms later at the equator than the singularity forms at the poles.

where A, B, C and D are constants depending on γ, a and b. For the choice of parameters $\gamma = 1/2,\ a = 1,\ b = \sqrt{5}$, for example, we find $A = -352/225,\ B = 8/9,\ C = 2/9$, and $D = 11/6$, from which it can be verified that σ_0 is everywhere positive. The MTS for this case is plotted in fig. 1.

The possibility exists, however, that our chosen MTS may not be the outermost one, and so would not be the outer boundary of a trapped region. (Examples where there exists an MTS which is not the outer boundary of a trapped region will be given in the next section.) In that case trapped surfaces would occur to the past of our MTS, and our argument about the inability to find trapped surfaces would not apply. We have investigated this issue numerically and have found that for at least some choices of parameters a, b, and γ, our MTS is indeed the outermost one.

We numerically examined the boundary value problem consisting of (20) together with $F'(\pi/2) = 0$ (to ensure reflection symmetry about the $\theta = \pi/2$ plane) and $F'(0) = 0$ (to ensure regularity at the poles). The singular nature of the differential equation at $\theta = 0$ prevents the direct application of the boundary conditions in the numerical routines. Instead, the equation is solved from $\theta = \pi/2$ toward $\theta = 0$, with initial conditions $F'(\pi/2) = 0$ and $F(\pi/2) = t_0$. We

then check to see if the boundary condition $F'(0) = 0$ is met.

For the choice of parameters $\gamma = 1/2,\ a = 1,\ b = \sqrt{5}$, the chosen MTS has $t_0 = 0.5$. For t_0 slightly less than 0.5, the solutions "curve more to the past" and $F'(\theta) \to \infty$ as $\theta \to 0$. As t_0 is further decreased, the solutions diverge even more rapidly as $\theta \to 0$, and the boundary condition $F'(0) = 0$ cannot be met. We conclude that the chosen MTS is the outermost one, and therefore no trapped surfaces exist outside of our chosen MTS.[1]

4 The Barrabès-Israel-Letelier model

In the Barrabès-Israel-Letelier model (BIL) model of prolate collapse, the initial shape is a cylinder of length L with hemispherical endcaps. As this shell collapses, it retains this form; this is one advantage of this model. The resulting singularity will be a "spindle" singularity with stronger, point singularities at the ends.

Again, we let (t, x, y, z) be Minkowski coordinates for the region interior to the shell. Since the singularity forms "simultaneously", it is convenient to label the hyperplane at which the singularity forms as $t = 0$, and choose t to *increase* towards the past (so that t is positive in the region of interest). We can write down $\mathcal{N}$ immediately in piecewise form:

$$\begin{array}{ll} x^\mu(t,z,\phi) = (-t, t\cos\phi, t\sin\phi, z) & \text{cylinder} \\ x^\mu(t,\theta,\phi) = (-t, t\sin\theta\cos\phi, t\sin\theta\sin\phi, t\cos\theta) & \text{right endcap} \\ x^\mu(t,\theta',\phi) = (-t, t\sin\theta'\cos\phi, t\sin\theta'\sin\phi, t\cos\theta') & \text{left endcap} \end{array} \tag{24}$$

$$z \in [-\frac{L}{2}, \frac{L}{2}]; \theta, \theta' \in [0, \frac{\pi}{2}]; \phi \in [0, 2\pi) \tag{25}$$

The pieces are matched by identifying $z = L/2$ with $\theta = \pi/2$, and $z = -L/2$ with $\theta' = \pi/2$.

The initial mass density is taken to be uniform over each component of the collapsing shell. Because the shape is retained during collapse, the density varies with time as

$$\sigma(t) = \begin{cases} \frac{M}{2\pi t L} & \text{cylinder} \\ \frac{m}{2\pi t^2} & \text{endcaps} \end{cases} \tag{26}$$

We will refer to M as "the total mass-energy of the cylindrical portion", and m as "the total mass-energy of each endcap". Note that, except for one value of t, the mass density is discontinuous where the cylinder meets the endcaps.

Again, we seek MTS's that have the same symmetries as $\mathcal{N}$, since the outermost MTS (which is the one of interest to us) must have these symmetries. Because of the reflection-symmetry of the MTS about $z = 0$, we focus on $z \geq 0$

[1] For t_0 slightly larger than 0.5, the solutions "curve more toward the future" and $F'(\theta) \to -\infty$ as $\theta \to 0$; these surfaces can be perturbed slightly to construct regular trapped surfaces bounded by our chosen MTS. When t_0 is slightly larger than 0.51, solutions run into the singularity before reaching $\theta = 0$. We therefore also conclude that there is no MTS forming at a later time than our chosen MTS.

and ignore the left endcap. Then, exploiting the axial symmetry, we can define a candidate 2-surface by the equations

$$t = \begin{cases} f(z) & \text{cylinder} \\ V(\theta) & \text{endcap} \end{cases} \tag{27}$$

It has been shown [9] that if f and V are to describe an MTS, they must satisfy [12]

$$-2f\frac{d^2 f}{dz^2} + 1 - \left(\frac{df}{dz}\right)^2 = \frac{8M}{L} \tag{28}$$

$$1 - \frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d}{d\theta}(\log V)\right) = \frac{4m}{V} \tag{29}$$

The MTS must be smooth at $z = 0$, regular at $\theta = 0$, but need only be C^1 at $z = \frac{L}{2}$ since $\mathcal{N}$ itself is only C^1 and the density is discontinuous there. Hence, we require

$$\frac{df}{dz}(0) = 0 \tag{30}$$

$$V(\frac{\pi}{2}) = f(\frac{L}{2}) \tag{31}$$

$$\frac{d}{d\theta}(\log V)(\frac{\pi}{2}) = -\frac{df}{dz}(\frac{L}{2}) \tag{32}$$

$$\frac{dV}{d\theta}(0) = 0 \tag{33}$$

The model is completely specified by the three parameters m, M and L. Since, as m, M and L are scaled together, the solution will scale with them, it is useful to define $\lambda = M/L$ and $\mu = m/M$. Then a given pair (λ, μ) defines an equivalence class of solutions with respect to overall change of mass/distance scale.

In the original presentation of the BIL model [7], it was demonstrated that for $\lambda < 1/8$, "the collapsing shell" (meaning the intersection of $\mathcal{N}$ with a flat hyperplane) is never a trapped or marginally-trapped surface, whereas for $\lambda > 1/8$, these intersection surfaces eventually become trapped. Our aim here is to investigate whether in the case $\lambda < 1/8$ trapped surfaces exist on the shell that do not lie on flat hyperplanes. We therefore restrict attention to $\lambda < 1/8$.

To investigate this issue, we proceed as follows. First, we fix M and L and solve eqs. (28) and (30) on the interval $[0, L/2]$, with $t_0 \equiv f(0)$ treated as an arbitrary free parameter. (Note that the parameter m does not enter these equations.) A unique solution to these equations exists [9] for all choices of $t_0 > 0$. The resulting value of f and its derivative at $z = L/2$ then generate the required initial conditions (31) and (32) for the system (29),(31),(32) and (33). This is an overdetermined system, so, in general, no solution will exist. However, in appendix A, we outline a proof (due to David Ross) that there always exists a unique choice of m such that a solution to these equations exists.

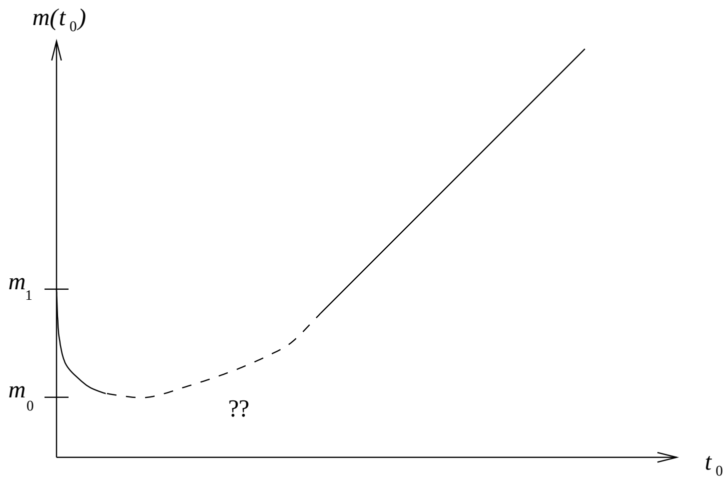

Figure 2: Qualitative behavior of m vs. t_0.

For each fixed M and L, this unique solution defines a continuous function $m(t_0)$. The asymptotic behavior of this function as $t_0 \to 0$ and as $t_0 \to \infty$ is analyzed in appendix B. As also noted there, it is not difficult to show that no solutions exist for $m = 0$. This fact together with the continuity and asymptotic behavior of $m(t_0)$ implies that there exists an $m_0 > 0$ such that $m(t_0) \geq m_0$ for all t_0. The resulting behavior of the function $m(t_0)$ is sketched in fig. 2.

From inspection of fig. 2, we see that at each fixed M and L, there exist three qualitatively distinct regimes with regard to solutions of the MTS equation. For $0 \leq m < m_0$, there do not exist any MTS's lying on $\mathcal{N}$. On the other hand, for $m_0 < m < m_1$, there exist at least two MTS's on $\mathcal{N}$. Finally, for m sufficiently large, there exists a unique MTS lying on $\mathcal{N}$.

In order to gain more insight into the behavior of the MTS's, we have numerically studied the solutions to the MTS equations for a wide range of the essential parameters λ and μ. For any fixed $\mu > 0$, we find that when λ is sufficiently close to $1/8$, there always exists a unique MTS (see fig. 3). This MTS is the outer boundary of a trapped region. As λ is decreased holding μ fixed (*i.e.,* as the shell is made more prolate), one encounters a critical value, $\lambda = \lambda_1$, beyond which a second MTS appears inside the first one, almost touching the singularity. As λ is decreased further, the outer MTS and inner MTS migrate towards each other, until, at a second critical value $\lambda = \lambda_0$, the two surfaces merge and then disappear (see fig. 4). For $\lambda < \lambda_0$, no marginally-trapped surfaces are found confined to the null hypersurface. These critical values λ_0 and λ_1 describe curves in the (λ, μ) plane, dividing parameter space into regions for which qualitatively different types of solutions occur (fig. 5).

The disappearance of trapped surfaces on $\mathcal{N}$ when $\lambda < \lambda_0(\mu)$ is in accord with the qualitative behavior one might have expected from the hoop conjecture. To investigate this more quantitatively, we interpret the "mass" appearing in

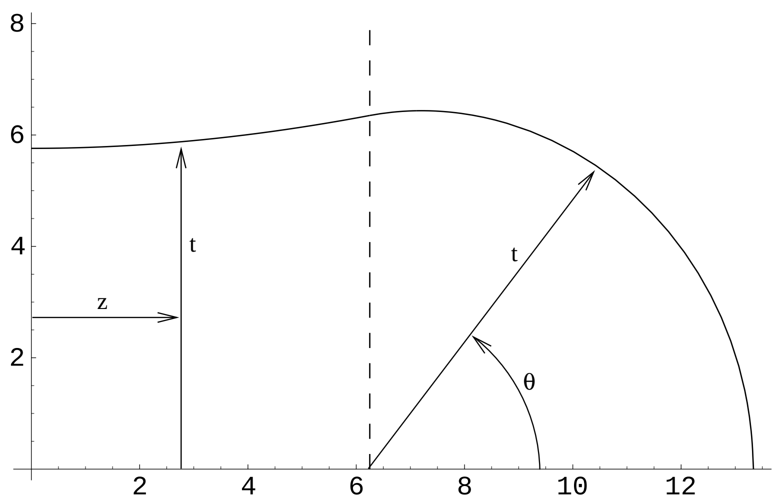

Figure 3: Interpreting the plots. We represent the MTS on $\mathcal{N}$ by projecting it onto a flat hyperplane in the Minkowski region and then suppressing the ϕ-coordinate. Shown is a typical MTS ($\mu = 2, \lambda = 0.08$). In the central region (to the left of the dotted line), the ordinate yields the time to the past of the singularity on null generator labelled by z on the cylindrical portion of the surface. In the end region (to the right of the dotted line), the radius (measured from the intersection of the dotted line with the abscissa) as a function of the polar angle θ yields the time to the past of the singularity on null generator labelled by θ on the hemispherical endcap.

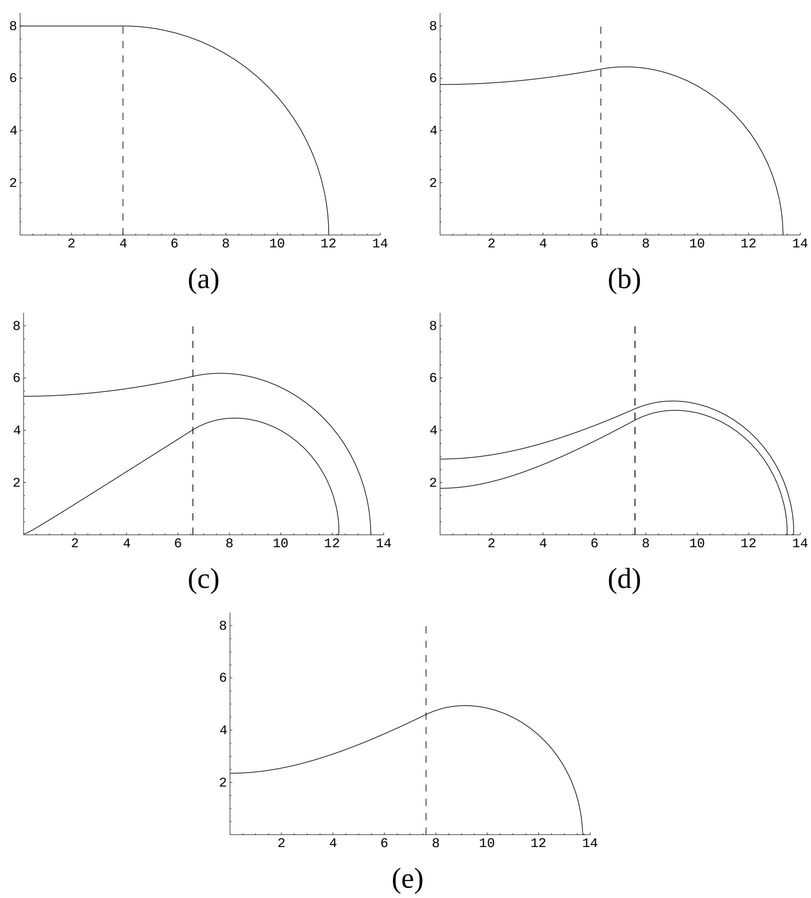

Figure 4: μ held fixed ($\mu = 2$), λ varied. (a) $\lambda = 0.125$; (b) $\lambda = 0.08$; (c) $\lambda = \lambda_1 = 0.076$; (d) $\lambda = 0.066$; (e) $\lambda = \lambda_0 = 0.0656$

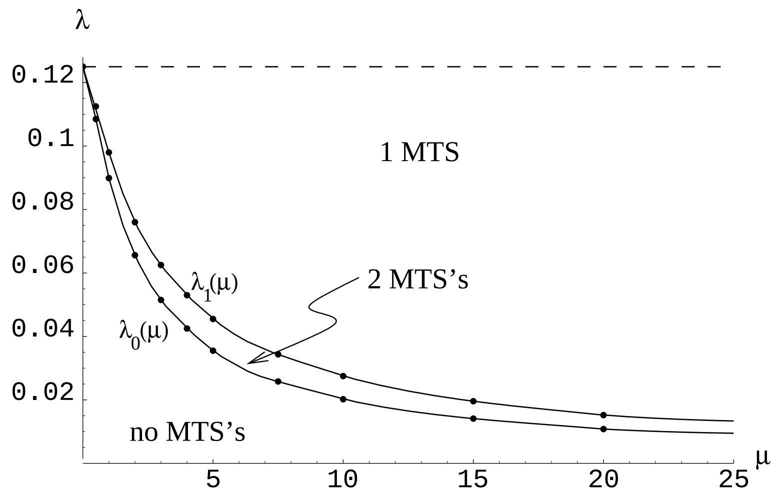

Figure 5: Phase diagram, λ vs. μ

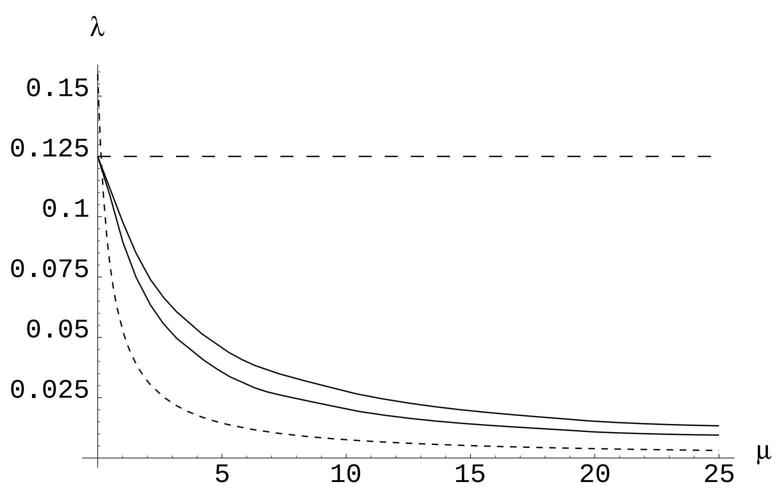

Figure 6: The same phase diagram as in fig. 5, with the function $\lambda = (2\pi(1+2\mu))^{-1}$ also plotted (dashed line).

the general statement of the hoop conjecture (see section 1) as being the total mass-energy, $M_{\rm tot} = M + 2m$, of the shell, and we take the "circumference" of a 2-surface on $\mathcal{N}$ to be the pole-to-pole circumference $2L + 4I$, where

$$I = \int_0^{\pi/2} V(\theta) d\theta \tag{34}$$

Since $C \geq 2L$ for all 2-surfaces on $\mathcal{N}$, with equality holding to arbitrarily good accuracy as one approaches the singularity, if this interpretation of the hoop conjecture is correct, one would expect that trapped surfaces will exist on $\mathcal{N}$ if and only if $2L \leq 4\pi(M + 2m)$, *i.e.* if and only if

$$\frac{1}{\lambda} \leq 2\pi(1 + 2\mu) \tag{35}$$

Thus, the hoop conjecture suggests that the curve defined by equality in eq. (35) should be the "dividing line" between the regimes where trapped surfaces do and do not occur on $\mathcal{N}$. This curve is plotted as a dashed line in fig. 6. It can be seen from the figure that this line lies remarkably close to the actual dividing line[2] $\lambda = \lambda_0(\mu)$.

Furthermore, even better quantitative agreement with expectations from the hoop conjecture is obtained if we consider the ratio $C/M_{\rm tot}$ for the "last" MTS, *i.e.* if we examine, as a function of μ, the quantity

$$\frac{C}{M_{\rm tot}} = \frac{2L + 4I}{M + 2m} = \frac{2 + \frac{4I}{L}}{(1 + 2\mu) \cdot \lambda_0(\mu)} \tag{36}$$

where I is given by eq. (34) above. In numerical solutions, we find that the ratio $C/M_{\rm tot}$ decreases monotonically with μ. Although it is as much as 30% greater than the 4π appearing in the statement of the hoop conjecture in the limit $\mu \to 0$, it is only about 2% less than 4π in the limit $\mu \to \infty$ (the limit of extreme prolateness).

Does the absence of trapped surfaces on $\mathcal{N}$ when $\lambda < \lambda_0(\mu)$ suggest that trapped surfaces are completely absent from the spacetime? If so, this could be viewed as evidence that the singularity produced by the collapse of the shell is naked in this regime. However, we do not believe that the absence of trapped surfaces on $\mathcal{N}$ when $\lambda < \lambda_0(\mu)$ provides strong evidence for the absence of trapped surfaces in the spacetime. When $\lambda = 1/8$ and $\mu > 0$, the outermost MTS is the intersection of $\mathcal{N}$ with a hyperplane at a time $t = t_{\rm MTS}$. The intersection of $\mathcal{N}$ with any hyperplane with $0 < t < t_{\rm MTS}$ is a trapped surface [7], so there is a large "trapped region" on $\mathcal{N}$. There is also a large trapped region to the future of $\mathcal{N}$. Namely, if $\mathcal{T}$ denotes a trapped surface on $\mathcal{N}$ and $\mathcal{L}$ denotes the null hypersurface generated by the outgoing null geodesics from $\mathcal{T}$, then (by the Raychaudhuri equation) every cross-section of $\mathcal{L}$ also is trapped.

[2] In comparison, the "isoperimetric inequality" $A \leq 16\pi M_{\rm tot}^2$ is satisfied by a wide margin for the MTS's occurring in the highly prolate case. Thus, the validity of this inequality [10] does not provide a "reason" why MTS's do not exist for $\lambda < \lambda_0(\mu)$.

As λ is decreased (with μ held fixed) the trapped region on $\mathcal{N}$ begins to "migrate away" from the singularity at $t = 0$ near the poles $(\theta, \theta' = 0)$. At $\lambda = \lambda_1(\mu)$, the trapped region on $\mathcal{N}$ begins to migrate away from the singularity near the equator $(z = 0)$ as well. For $\lambda_0(\mu) < \lambda < \lambda_1(\mu)$, the trapped region on $\mathcal{N}$ presumably is the region bounded by the two MTS's present on $\mathcal{N}$ (see fig. 4d). However, trapped surfaces also must continue to exist in the region to the future of $\mathcal{N}$ swept out by outgoing null geodesics orthogonal to the trapped surfaces on $\mathcal{N}$. When $\lambda = \lambda_0$, there is a single MTS, $\mathcal{M}$, on $\mathcal{N}$, but no trapped surfaces on $\mathcal{N}$. However, since the shear of the outgoing null geodesics orthogonal to $\mathcal{M}$ is nonvanishing, there must continue to exist trapped surfaces to the future of $\mathcal{N}$. Thus, the disappearance of trapped surfaces on $\mathcal{N}$ at $\lambda = \lambda_0$ does not signal the disappearance of trapped surfaces from the spacetime, but rather their migration to the future of $\mathcal{N}$. However, the issue of whether or not trapped surfaces continue to be present as $\lambda \to 0$ cannot be investigated until we have a much more detailed knowledge of the spacetime exterior to $\mathcal{N}$.

Acknowledgements

We wish to thank David Ross for providing us with the existence and uniqueness proofs sketched in Appendix A. This research was supported in part by NSF grant PHY 95-14726 to the University of Chicago. Two of us (K.P.T. and R.M.W.) also wish to thank the Erwin Schrödinger Institute for hospitality during the summer of 1994, where some of the ideas for this research project arose.

A Existence and uniqueness of solutions

In this appendix, we outline a proof that in the BIL model, given any $t_0 > 0$, there exists a unique value of m yielding an MTS on $\mathcal{N}$ with $f(0) = t_0$. (That is, we prove that the function $m(t_0)$ described in the text is well-defined for all $t_0 > 0$.) The key ideas in the existence and uniqueness proof are due to David Ross.

We fix M, L, and t_0, and leave m to be determined. We wish to find a solution of the following system of equations:

$$-2f\frac{d^2f}{dz^2} + 1 - \left(\frac{df}{dz}\right)^2 = \frac{8M}{L} \tag{37}$$

$$\frac{df}{dz}(0) = 0 \tag{38}$$

$$f(0) = t_0 \tag{39}$$

$$1 - \frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d}{d\theta}(\log V)\right) = \frac{4m}{V} \tag{40}$$

$$V(\frac{\pi}{2}) = f(\frac{L}{2}) \tag{41}$$

$$\frac{d}{d\theta}(\log V)(\frac{\pi}{2}) = -\frac{df}{dz}(\frac{L}{2}) \tag{42}$$

$$\frac{dV}{d\theta}(0) = 0 \tag{43}$$

One can derive the following implicit solution [9] for the first three equations (37-39)

$$\frac{2\alpha z}{t_0} = \frac{1}{w-1} + \frac{1}{w+1} - \log\left(\frac{w-1}{w+1}\right) \tag{44}$$

$$w = \sqrt{\frac{f}{f-t_0}} \tag{45}$$

where

$$\alpha = \sqrt{1 - \frac{8M}{L}} \tag{46}$$

This solution generates certain values $f(L/2) > 0$ and $0 < df/dz(L/2) < 1$; taking these values as given (determined by our choices of M, L, and t_0), we focus on (40-43). This system of equations, a 2nd-order ODE with three boundary conditions, would be overdetermined if we also specified m. However, we instead recast eqs. (40), (42), and (43) in terms of the "scale-invariant" function $\hat{V} = V/m$, which obeys

$$1 - \frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d}{d\theta}(\log\hat{V})\right) = \frac{4}{\hat{V}} \tag{47}$$

$$\frac{d}{d\theta}(\log\hat{V})(\frac{\pi}{2}) = -\frac{df}{dz}(\frac{L}{2}) \tag{48}$$

$$\frac{d\hat{V}}{d\theta}(0) = 0 \tag{49}$$

These equations do not involve m. The remaining boundary condition (41) is then satisfied if and only if m is chosen to be

$$m = \frac{f(L/2)}{\hat{V}(\frac{\pi}{2})}. \tag{50}$$

It follows that if there exists a unique solution to (47-49), then there exists a unique value of m yielding an MTS with the chosen t_0.

Now write $k = -df/dz(L/2)$, let primes denote derivatives with respect to θ, and define $W \equiv (\log\hat{V})'$. This gives

$$1 - \frac{1}{\sin\theta}(W\sin\theta)' = \frac{4}{\hat{V}} \tag{51}$$

$$\hat{V}' = W\hat{V} \tag{52}$$

$$W(\frac{\pi}{2}) = k \tag{53}$$

$$W(0) = 0 \tag{54}$$

In order to prove the existence and uniqueness of solutions to this boundary-value problem, we convert it to an initial-value problem and use a "shooting argument", *i.e.* we consider solutions to (51), (52), (54) and

$$\hat{V}(0) = \hat{V}_0 > 0 \tag{55}$$

and then demonstrate that one can choose $\hat{V}_0$ uniquely in order to satisfy the boundary condition (53). We will first prove local existence and uniqueness of solutions to (51), (52), (54) and (55) around $\theta = 0$, then show global existence and uniqueness on the interval $[0, \pi/2]$ for at least some choices of $\hat{V}_0$, and finally argue that (53) can always be satisfied.

To begin, we note that (51) and (54) are equivalent to the integral equation

$$W = \frac{1}{\sin\theta} \int_0^\theta \left(1 - \frac{4}{\hat{V}(t)}\right) \sin t \; dt \tag{56}$$

whereas eqs. (52) and (55) are equivalent to

$$\hat{V}(\theta) = \hat{V}_0 + \int_0^\theta W(t)\hat{V}(t) \; dt \tag{57}$$

To establish local existence of solutions to (56) and (57), we use a variant of "Picard iteration". Consider the iteration scheme given by

$$W_j(\theta) = \frac{1}{\sin\theta} \int_0^\theta \left(1 - \frac{4}{\hat{V}_{j-1}(t)}\right) \sin t \; dt \tag{58}$$

$$\hat{V}_j(\theta) = \hat{V}_0 + \int_0^\theta W_{j-1}(t)\hat{V}_{j-1}(t) \; dt \tag{59}$$

with initial "guesses" $\hat{V}_{j=1} = \hat{V}_0$ and $W_{j=1} = 0$. It can be shown, by demonstrating that the W_j's and $\hat{V}_j$'s are bounded in some small interval beginning at $\theta = 0$, that the iterates converge uniformly as

$$|\hat{V}_j - \hat{V}_{j-1}| < \frac{(C\theta)^j}{j!} \tag{60}$$

$$|W_j - W_{j-1}| < \frac{(C\theta)^j}{j!} \tag{61}$$

where C is a constant. Because the convergence is uniform, the limit functions $\hat{V} = \lim \hat{V}_j$ and $W = \lim W_j$ satisfy (56) and (57), and so constitute a (local) solution to the initial-value problem.

Local uniqueness is proved similarly. Let $\hat{V}_1, W_1$ and $\hat{V}_2, W_2$ both be solutions of the initial-value problem. Then, again using the fact that both of these

solutions are bounded on a sufficiently small interval starting at $\theta = 0$, one can show that for all positive integers j,

$$|\hat{V}_1 - \hat{V}_2| < P\frac{(2Q)^j \theta^j}{j!} \tag{62}$$

$$|\hat{W}_1 - \hat{W}_2| < P\frac{(2Q)^j \theta^j}{j!} \tag{63}$$

where P and Q are constants. Therefore, $V_1 = V_2$ and $W_1 = W_2$, and solutions of the initial-value problem are unique near $\theta = 0$.

By a standard theorem in the theory of ODE's [13], then, this unique local solution to eqs. (56) and (57) can be extended away from $\theta = 0$ until it becomes unbounded. It remains to be shown that $\hat{V}_0$ can be chosen so that the solution can be extended all the way to $\theta = \pi/2$ and satisfies $W(\frac{\pi}{2}) = k$. It is not difficult to prove that, for fixed $\theta > 0$, $\hat{V}$ and W are strictly increasing functions of $\hat{V}_0$, so if we can find a $\hat{V}_0$ which yields the correct matching condition, this $\hat{V}_0$ will be unique.

The analysis of global existence of solutions on the interval $[0, \pi/2]$ naturally divides into the following three cases.

- $\hat{V}_0 > 4$. Then $\hat{V}$ and W are strictly increasing functions of θ. Furthermore, $\hat{V}$ and W are bounded on the entire interval $[0, \pi/2]$, according to

$$\hat{V}_0 \leq \hat{V}(\theta) \leq \hat{V}_0 \exp\left(\int_0^\theta \frac{1-\cos t}{\sin t}\, dt\right) \tag{64}$$

$$0 \leq W(\theta) < \frac{1-\cos\theta}{\sin\theta} \tag{65}$$

  with equalities occurring only at $\theta = 0$. As $\hat{V}_0 \to \infty$, $W(\pi/2) \to 1$, while as $\hat{V}_0 \to 4$, $W(\pi/2) \to 0$. By continuous dependence of solutions on initial data, it follows that all values of $W(\pi/2)$ satisfying $0 < W(\pi/2) < 1$ can be achieved.

- $\hat{V}_0 = 4$. Then $\hat{V}(\theta) = 4$ and $W(\theta) = 0$.

- $\hat{V}_0 < 4$. Then $W < 0$ for $\theta > 0$, and, for fixed $\hat{V}_0$, $\hat{V}(\hat{V}_0, \theta)$ is a strictly decreasing function of θ. (Recall that for fixed θ, $\hat{V}(\hat{V}_0, \theta)$ is also a strictly increasing function of $\hat{V}_0$.) In this case, it is possible that for some $\hat{V}_0$, the solution "blows up" ($\hat{V} \to 0$ and $W \to -\infty$) before reaching $\theta = \pi/2$, in which case a global solution does not exist. (It is not difficult to show that this is the only way in which a global solution can fail to exist.) We wish to show that given any $k \in (-\infty, 0)$, $\hat{V}_0$ can nevertheless be chosen to yield a global solution with $W(\pi/2) = k$. First, we note that since $\hat{V} = 4$ is a global solution, it follows from continuous dependence of solutions on initial data that there exist global solutions on the interval $[0, \pi/2]$ for some $\hat{V}_0 < 4$. Let c be the greatest lower bound of the set of $\hat{V}_0$

for which global solutions exist. Then, from the monotonicity properties of $\hat{V}$, it follows that global solutions exist for all $\hat{V}_0 \in (c, 4)$. Let p be the greatest lower bound of $\hat{V}(\pi/2)$ for these global solutions (so that, by monotonicity, as $\hat{V}_0 \to c$, $\hat{V}(\hat{V}_0, \pi/2) \to p$). Then there exist three possibilities: (i) If $p = 0$, then it can be shown that as $\hat{V}(\hat{V}_0, \pi/2) \to 0$, $W(\pi/2) \to -\infty$, Therefore, by continuous dependence of solutions on initial data, any value $W(\pi/2) \in (-\infty, 0)$ can be obtained by some choice of $\hat{V}_0 \in (c, 4)$. (ii) If $p > 0$ and $c = 0$, then we contradict the fact that $\hat{V}(\theta)$ is strictly decreasing in θ. (iii) If $p > 0$ and $c > 0$, then one can show, using continuous dependence of solutions on initial conditions and the fact that solutions can fail to exist globally only by $\hat{V}$ approaching 0 at some $\theta \le \pi/2$, that there exists a global solution with $\hat{V}_0 = c$ and $\hat{V}(\pi/2) = p$. However, again using continuous dependence of solutions on initial data, it then follows that there exist global solutions with $\hat{V}_0 < c$. Thus we obtain a contradiction. Consequently, only case (i) is possible.

In summary, for any $\hat{V}_0 > c$, we can obtain a global solution to the initial-value problem on the interval $[0, \pi/2]$, and we can uniquely choose $\hat{V}_0$ in this domain in order to achieve any $W(\pi/2) < 1$. (Recall that we only needed $-1 < W(\pi/2) < 0$.) Therefore, there exists a unique solution to the boundary-value problem (47-49), and consequently a unique value of m, determined by (50), yielding an MTS with the chosen t_0.

B Limiting behavior of $m(t_0)$

Given M and L, we showed in appendix A that m is determined by a choice of t_0. This dependence is mediated by the "matching conditions" between f and V, the center and end solutions. We write

$$X(t_0) = f(t_0, z = L/2) \tag{66}$$

$$Y(t_0) = \frac{\partial f}{\partial z}(t_0, z = L/2) \tag{67}$$

where $f(t_0, z)$ denotes the center solution, $f(z)$ (see eqs. (44),(45)), with $f(0) = t_0$. It is convenient to view X and Y as independent variables and to view m as a function of X and Y,

$$m = m(X, Y) \tag{68}$$

We may then write

$$\frac{dm}{dt_0} = \frac{\partial m}{\partial X}\frac{dX}{dt_0} + \frac{\partial m}{\partial Y}\frac{dY}{dt_0} \tag{69}$$

In this appendix, we derive the limiting behavior as $t_0 \to 0$ and as $t_0 \to \infty$ of each of the terms on the RHS of this equation, and so discover the limiting behavior of $m(t_0)$.

We first examine how m changes with X when Y is held fixed. Since m is determined by solving the system of equations (47-49), and these equations are

independent of X, it follows that if X is changed, keeping Y fixed, we obtain the same solution $\hat{V}$. From eq. (50), we therefore obtain

$$\frac{\partial m}{\partial X} = \frac{1}{\hat{V}(\pi/2)} \tag{70}$$

Next, we change Y while holding X fixed. Since we found in the previous appendix that $\hat{V}(\pi/2)$ and $W(\pi/2)$ are strictly monotonically increasing functions of $\hat{V}_0$, an increase in Y results in a decrease in $W(\pi/2)$, a decrease in $\hat{V}_0$, and a decrease in $\hat{V}(\pi/2)$. By eq. (50), the required m must increase, and so

$$\frac{\partial m}{\partial Y} > 0 \tag{71}$$

For the remaining terms, we need to refer to the implicit solution (44) and (45) for $f(z)$, as well as the first integral of eq. (37)

$$\frac{\partial f}{\partial z}(t_0, z) = \alpha\sqrt{1 - \frac{t_0}{f(t_0, z)}} \tag{72}$$

By taking the derivatives with respect to t_0 of the implicit solution (44) and the first integral (72) we obtain, respectively,

$$\frac{dX}{dt_0} = \frac{X}{t_0} - \frac{\alpha L}{2t_0}\sqrt{1 - \frac{t_0}{X}} \tag{73}$$

$$\frac{dY}{dt_0} = -\frac{\alpha^2 L}{4X^2} \tag{74}$$

Although our knowledge of dm/dt_0 is limited by the fact that we have only qualitative knowledge of $\partial m/\partial Y$, eq. (71), we can nevertheless derive its exact behavior in the limits $t_0 \to 0$ and $t_0 \to \infty$. When $t_0 \ll L$, the implicit solution for f yields, to lowest order,

$$X \approx \frac{\alpha L}{2} + \frac{t_0}{2}\log\left(\frac{t_0}{\alpha L}\right) \tag{75}$$

implying

$$\frac{dX}{dt_0} \sim \log\left(\frac{t_0}{\alpha L}\right) \to -\infty \text{ as } t_0 \to 0 \tag{76}$$

On the other hand, since $0 < \partial f/\partial z(t_0, z) < \alpha$,

$$t_0 < X < t_0 + \alpha L/2 \tag{77}$$

Then eq. (73) gives

$$\frac{dX}{dt_0} \to 1 \text{ as } t_0 \to \infty \tag{78}$$

Since, by the above, $X \to \alpha L/2$ as $t_0 \to 0$ and $X \to \infty$ as $t_0 \to \infty$, we have from eq. (74)

$$\frac{dY}{dt_0} \to \begin{cases} -\frac{1}{L} & \text{as } t_0 \to 0 \\ 0 & \text{as } t_0 \to \infty \end{cases} \tag{79}$$

According to eq. (72),

$$Y \to \begin{cases} \alpha & \text{as } t_0 \to 0 \\ 0 & \text{as } t_0 \to \infty \end{cases} \tag{80}$$

Now, applying a result of the previous appendix, $\hat{V}(\pi/2)$ approaches some value between 0 and 4 in the first limit, and $\hat{V}(\pi/2)$ goes to 4 in the second limit. Thus

$$\frac{\partial m}{\partial X} \to \begin{cases} \text{const.} > \frac{1}{4} & \text{as } t_0 \to 0 \\ \frac{1}{4} & \text{as } t_0 \to \infty \end{cases} \tag{81}$$

Returning to eq. (69), we have that

$$\frac{dm}{dt_0} \to -\infty \text{ as } t_0 \to 0 \tag{82}$$

whereas m approaches a finite limiting value, m_1, as $t_0 \to 0$ (since there are no special difficulties encountered in solving (47-49) when $\partial f/\partial z(L/2) = \alpha$). In the other limit, we have simply

$$\frac{dm}{dt_0} \to \frac{1}{4} \text{ as } t_0 \to \infty \tag{83}$$

Finally, note that when $m = 0$, we can explicitly solve for $V(\theta)$, finding that $(\log V)'(\pi/2) = 1$. However, it is impossible to choose t_0 in order to satisfy the corresponding matching condition $\partial f/\partial z(L/2) = -1$. Thus there exists no MTS confined to the null hypersurface $\mathcal{N}$ when $m = 0$.

The above results imply that $m(t_0)$ has the behavior described in section 4 (see figure 2 above).

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